

RIGGED MODULES I: MODULES OVER DUAL OPERATOR ALGEBRAS AND THE PICARD GROUP

DAVID P. BLECHER AND UPASANA KASHYAP

ABSTRACT. In a previous paper we generalized the theory of W^* -modules to the setting of modules over nonselfadjoint dual operator algebras, obtaining the class of weak*-rigged modules. At that time we promised a forthcoming paper devoted to other aspects of the theory. We fulfill this promise in the present work and its sequel “Rigged modules II”, giving many new results about weak*-rigged modules and their tensor products. We also discuss the Picard group of weak* closed subalgebras of a commutative algebra. For example, we compute the weak Picard group of $H^\infty(\mathbb{D})$, and prove that for a weak* closed function algebra A , the weak Picard group is a semidirect product of the automorphism group of A , and the subgroup consisting of symmetric equivalence bimodules.

1. INTRODUCTION AND NOTATION

The most important class of modules over a C^* -algebra M are the Hilbert C^* -modules: the modules possessing an M -valued inner product satisfying the natural list of axioms (see [19] or [9, Chapter 8]). A W^* -module is a Hilbert C^* -module over a von Neumann algebra which is selfdual (that is, it satisfies the module variant of the fact from Hilbert space theory that every continuous functional is given by inner product with a fixed vector, see e.g. [21, 4]), or equivalently which has a Banach space predual (see e.g. [10, Corollary 3.5]). A *dual operator algebra* is a unital weak* closed algebra of operators on a Hilbert space (which is not necessarily selfadjoint). One may think of a dual operator algebra as a nonselfadjoint analogue of a von Neumann algebra. The *weak*-rigged* or *w*-rigged modules*, introduced in [7] (see also [8, 17]), are a generalization of W^* -modules to the setting of modules over a (nonselfadjoint) dual operator algebra. In [7] we generalized basic aspects of the theory of W^* -modules, and this may be seen also as the weak* variant of the theory of rigged modules from [3] (see also [11]). In that paper we promised that some other aspects of the theory of weak*-rigged modules would be presented elsewhere. Since that time is now long overdue, and since there has been some recent interest in rigged modules and related objects (see e.g. [20] or the survey [14] of Eleftherakis’ work), we follow through on our promise here and in the sequel [5].

The present paper and its sequel consists of several topics and results about weak*-rigged modules, mainly concerning their tensor products. For example following the route in [1] we study the Picard group of weak* closed subalgebras of a commutative algebra. For a weak* closed function algebra A , the weak Picard group $Pic_w(A)$ is a semidirect product of $\text{Aut}(A)$, the automorphism group of A ,

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and the subgroup of $Pic_w(A)$ consisting of symmetric equivalence bimodules. In particular, we show that the weak Picard group of $H^\infty(\mathbb{D})$ is isomorphic to the group of conformal automorphisms of the disk.

We will use the notation from [6, 7, 18], and perspectives from [2]. We will assume that the reader is familiar with basic notions from operator space theory which may be found in any current text on that subject (such as [13, 9]). The reader may consult [9] as a reference for any other unexplained terms here. We also assume that the reader is familiar with basic Banach space and operator space duality principles, e.g., the Krein-Smulian Theorem (see e.g., Section 1.4, 1.6, Appendix A.2 in [9]). We often abbreviate ‘weak*’ to ‘ w^* ’. We use the letters H, K for Hilbert spaces. By a nonselfadjoint analogue of Sakai’s theorem (see e.g. Section 2.7 in [9]), a dual operator algebra M is characterized as a unital operator algebra which is also a dual operator space. By a *normal morphism* we shall always mean a unital weak* continuous completely contractive homomorphism on a dual operator algebra. A concrete dual operator M - N -bimodule is a weak* closed subspace X of $B(K, H)$ such that $\theta(M)X\pi(N) \subset X$, where θ and π are normal morphism representations of M and N on H and K respectively. An abstract dual operator M - N -bimodule is defined to be a nondegenerate operator M - N -bimodule X , which is also a dual operator space, such that the module actions are separately weak* continuous. Such spaces can be represented completely isometrically as concrete dual operator bimodules, and in fact this can be done under even weaker hypotheses (see e.g. [9, 10, 12]) and similarly for one-sided modules (the case M or N equals \mathbb{C}). We use standard notation for module mapping spaces; e.g. $CB(X, N)_N$ (resp. $CB^\sigma(X, N)_N$) are the completely bounded (resp. and weak* continuous) right N -module maps from X to N . We often use the *normal module Haagerup tensor product* $\otimes_M^{\sigma h}$, and its universal property from [15], which loosely says that it ‘linearizes’ completely contractive M -balanced separately weak* continuous bilinear maps (balanced means that $u(xa, y) = u(x, ay)$ for $a \in M$). We assume that the reader is familiar with the notation and facts about this tensor product from [6, Section 2]. For any operator space X we write $C_n(X)$ for the column space of $n \times 1$ matrices with entries in X , with its canonical norm from operator space theory.

Definition 1.1. [7] *Suppose that Y is a dual operator space and a right module over a dual operator algebra M . Suppose that there exists a net of positive integers $(n(\alpha))$, and w^* -continuous completely contractive M -module maps $\phi_\alpha : Y \rightarrow C_{n(\alpha)}(M)$ and $\psi_\alpha : C_{n(\alpha)}(M) \rightarrow Y$, with $\psi_\alpha(\phi_\alpha(y)) \rightarrow y$ in the weak* topology on Y , for all $y \in Y$. Then we say that Y is a right w^* -rigged module (or weak*-rigged module) over M .*

As on p. 348 of [7], the operator space structure of a w^* -rigged module Y over M is determined by

$$\|[y_{ij}]\|_{M_n(Y)} = \sup_{\alpha} \|[\phi_\alpha(y_{ij})]\|, \quad [y_{ij}] \in M_n(Y).$$

The following seems not to have been proved in the development in Section 2 and the start of Section 3 in [7] but seemingly assumed there.

Lemma 1.2. *A right w^* -rigged module over a dual operator algebra M is a dual operator M -module.*

Proof. Let Y be the w^* -rigged module, and let $\phi_\alpha, \psi_\alpha, n_\alpha$ be as in Definition 1.1. We need to show that the module action $Y \times M \rightarrow Y$ is separately weak* continuous.

Given a bounded net $m_t \rightarrow m$ weak* in M , and $y \in Y$, suppose that a subnet $ym_{t_\nu} \rightarrow y'$ weak* in Y . Then $\phi_\alpha(\psi_\alpha(ym_{t_\nu})) = \phi_\alpha(\psi_\alpha(y)m_{t_\nu})$ weak* converges both to $\phi_\alpha(\psi_\alpha(y)m) = \phi_\alpha(\psi_\alpha(ym))$ and $\phi_\alpha(\psi_\alpha(y'))$ with ν , for every α . Hence $y' = ym$, so that by topology $ym_t \rightarrow ym$ weak*. So the map $m \mapsto ym$ is weak* continuous by the Krein-Smulian theorem. That each $m \in M$ acts weak* continuously on Y follows from e.g. [9, Corollary 4.7.7]. \square

Every right w^* -rigged module Y over M gives rise to a canonical left w^* -rigged M -module \tilde{Y} , and a pairing $(\cdot, \cdot) : \tilde{Y} \times Y \rightarrow M$ (see [7]). Indeed \tilde{Y} turns out to be completely isometric to $CB^\sigma(Y, M)_M$ as dual operator M -modules, together with its canonical pairing with Y . We also have $\tilde{\tilde{Y}} = Y$. The morphisms between w^* -rigged M -modules are the *adjointable* M -module maps; these are the M -module maps $T : Y_1 \rightarrow Y_2$ for which there exists a $S : \tilde{Y}_2 \rightarrow \tilde{Y}_1$ with $(x, T(y)) = (S(x), y)$ for all $x \in \tilde{Y}_2, y \in Y_1$. These turn out to coincide with the weak* continuous completely bounded M -module maps (see [7, Proposition 3.4]). We often write $\mathbb{B}(Z, W)$ for the weak* continuous completely bounded M -module maps from a w^* -rigged M -module Z into a dual operator M -module W , with as usual $\mathbb{B}(Z) = \mathbb{B}(Z, Z)$.

In [7], Section 4 we gave several equivalent definitions of w^* -rigged modules (an additional note that seems to be needed to connect the definitions may be found mentioned in the proof of [5, Theorem 2.3]). From some of these it is clear that every weak* Morita equivalence N - M -bimodule Y in the following sense, is a w^* -rigged right M -module and a w^* -rigged left N -module, and its ‘dual module’ \tilde{Y} and pairing $(\cdot, \cdot) : \tilde{Y} \times Y \rightarrow M$ may be taken to be the X below, and pairing corresponding to the first \cong below.

Definition 1.3. *We say that two dual operator algebras M and N are weak* Morita equivalent, if there exist a dual operator M - N -bimodule X , and a dual operator N - M -bimodule Y , such that $M \cong X \otimes_N^{\sigma^h} Y$ as dual operator M -bimodules (that is, completely isometrically, w^* -homeomorphically, and also as M -bimodules), and similarly $N \cong Y \otimes_M^{\sigma^h} X$ as dual operator N -bimodules. We call (M, N, X, Y) a weak* Morita context in this case.*

2. RIGGED AND WEAK*-RIGGED MODULES AND THEIR TENSOR PRODUCT

2.1. Interior tensor product of weak* rigged modules. Suppose that Y is a right w^* -rigged module over a dual operator algebra M and, that Z is a right w^* -rigged module over a dual operator algebra N , and $\theta : M \rightarrow \mathbb{B}(Z)$ is a normal morphism. Because Z is a left operator module $\mathbb{B}(Z)$ -module (see p. 349 in [7]), Z becomes an essential left dual operator module over M under the action $m \cdot z = \theta(m)z$. In this case we say Z is a right M - N -correspondence. We form the normal module Haagerup tensor product $Y \otimes_M^{\sigma^h} Z$ which we also write as $Y \otimes_\theta Z$. By 3.3 in [7] this a right w^* -rigged module over N . We call $Y \otimes_\theta Z$ the *interior tensor product* of w^* -rigged modules. By 3.3 in [7] we have $\widetilde{Y \otimes_\theta Z} \cong \tilde{Z} \otimes_\theta \tilde{Y}$ with N -valued pairing

$$(w \otimes x, y \otimes z)_N = (w, \theta((x, y)_M)z)_N$$

of $\tilde{Z} \otimes_\theta \tilde{Y}$ with $Y \otimes_M^{\sigma^h} Z$.

The w^* -rigged interior tensor product is associative. That is, if A , B , and C are dual operator algebras, if X is a right w^* -rigged module over A , Y is a right w^* -rigged module over B , and Z is a right w^* -rigged module over C , and if $\theta : A \rightarrow \mathbb{B}(Y)$ and $\rho : B \rightarrow \mathbb{B}(Z)$ are normal morphisms, then $(X \otimes_\theta Y) \otimes_\rho Z \cong$

$X \otimes_\theta (Y \otimes_\rho Z)$ completely isometrically and w^* -homeomorphically. This follows from the associativity of the normal module Haagerup tensor product (see Proposition 2.9 in [6]).

Let Y be a right w^* -rigged module over a dual operator algebra M . If N is a W^* -algebra and if Z is a right W^* -module over N and $\theta : M \rightarrow \mathbb{B}(Z)$ is a normal morphism, then as we said earlier, $Y \otimes_\theta Z$ is a right w^* -rigged module over N . Since N is a W^* -algebra it follows from Theorem 2.5 in [7] that $Y \otimes_\theta Z$ is a right W^* -module over N . An important special case of this is the W^* -dilation. If $Z = N$ is a von Neumann algebra containing (a weak* homeomorphic completely isometrically isomorphic copy of) M , with the same identity element, and $\theta : M \rightarrow N$ is the inclusion, then $Y \otimes_\theta N$ is called the W^* -dilation of Y (see [18, 6, 7]). The following is an application of the W^* -dilation to Morita equivalence.

Theorem 2.1. *Suppose that Y is a right w^* -rigged module over a dual operator algebra M . Suppose that N is a von Neumann algebra containing M as a weak* closed subalgebra (with the same identity), and that \mathcal{R} is the weak* closed ideal in N generated by the range of the pairing $\tilde{Y} \times Y \rightarrow M$. Let $\theta : M \rightarrow N$ be the inclusion, which also induces a map $\pi : M \rightarrow B_{\mathcal{R}}(\mathcal{R}) \cong \mathcal{R}$ via the canonical left action. Then $Z = Y \otimes_\theta N = Y \otimes_\pi \mathcal{R}$ is a von Neumann algebraic Morita equivalence bimodule (in the sense of Rieffel [22]), implementing a von Neumann algebraic Morita equivalence between \mathcal{R} and $\mathbb{B}(Z) = Y \otimes_\pi \mathcal{R} \otimes_\pi \tilde{Y}$. In particular, if $W^*(M)$ is a von Neumann algebra generated by (a weak* homeomorphic completely isometrically isomorphic copy of) M , and if the range of the pairing $\tilde{Y} \times Y \rightarrow M$ is weak* dense in M , then $W^*(M)$ is Morita equivalent (in the sense of Rieffel) to the W^* -algebra $\mathbb{B}(Y \otimes_\theta W^*(M)) = Y \otimes_\theta W^*(M) \otimes_\theta \tilde{Y}$.*

Proof. By the lines above the theorem, $Z = Y \otimes_\theta N$ is a right W^* -module over N . That $Y \otimes_\theta N = Y \otimes_\pi \mathcal{R}$ may be seen for example by [7, Theorem 3.5] (see Lemma 2.5 below), since any weak* continuous left M -module map $\tilde{Y} \rightarrow N$ necessarily takes values in the ideal \mathcal{R} since terms of form $(x, y)x'$ are weak* dense in \tilde{Y} , for $x, x' \in \tilde{Y}, y \in Y$. By Corollary 8.5.5 in [9], $\mathbb{B}(Z)$ is a W^* -algebra. Consider the canonical pairing

$$\tilde{Z} \times Z \cong (N \otimes_\theta \tilde{Y}) \times (Y \otimes_\theta N) \rightarrow N.$$

The w^* -closure of the range of this pairing is a weak* closed two sided ideal in N . It is easy to see that this ideal is \mathcal{R} , since the terms of the form $a(xy)b$ are contained in the range of the above pairing for all $(x, y) \in \tilde{Y} \times Y$ and $a, b \in N$. By [22] (or 8.5.14 in [9]), \mathcal{R} and $\mathbb{B}(Z)$ are Morita equivalent in the sense of Rieffel. That

$$\mathbb{B}(Z) = Y \otimes_M^{\sigma h} (\mathcal{R} \otimes_{\mathcal{R}}^{\sigma h} \mathcal{R}) \otimes_M^{\sigma h} \tilde{Y} = Y \otimes_\theta \mathcal{R} \otimes_\theta \tilde{Y}$$

follows from the second paragraph on p. 357 of [7].

Since $W^*(M)$ is a W^* -algebra generated by M , and it is well known that the weak* closed ideals of W^* -algebras are selfadjoint, these together imply that the weak* closure of the range of the above pairing is all of $W^*(M)$. By 8.5.14 in [9], $W^*(M)$ and $\mathbb{B}(Y \otimes_\theta W^*(M))$ are Morita equivalent in the sense of Rieffel. \square

2.2. Functorial properties. Note that $M_{m,n}(\mathbb{B}(Y_1, Y_2))$ has two natural norms: the operator space one, coming from $CB(Y_1, M_{m,n}(Y_2))$, or the norm coming from $CB(C_n(Y_1), C_m(Y_2))$ via the identification of a matrix $[f_{ij}] \in M_{m,n}(CB(Y_1, Y_2))$ with the map $[y_j] \mapsto [\sum_j f_{ij}(y_j)]$. The next result asserts that these norms on $M_{m,n}(\mathbb{B}(Y_1, Y_2))$ are the same. We will write this norm as $\|[f_{ij}]\|_{cb}$.

Lemma 2.2. ([7, Corollary 3.6]) *Suppose that Y_1 and Y_2 are right w^* -rigged modules over a dual operator algebra M . For each $m, n \in \mathbb{N}$ we have $M_{m,n}(\mathbb{B}(Y, Z)) \cong \mathbb{B}(C_n(Y), C_m(Z))$ completely isometrically.*

The interior tensor product of w^* -rigged modules is functorial:

Proposition 2.3. *Suppose that Y_1 and Y_2 are right w^* -rigged modules over a dual operator algebra M , that Z_1 and Z_2 are right M - N -correspondences for a dual operator algebra N . Write the left action on Z_k (abusively) as $\theta : M \rightarrow \mathbb{B}(Z_k)_N$. If $f = [f_{ij}] \in M_{m,n}(\mathbb{B}(Y_1, Y_2)_M)$, and if $g = [g_{kl}] \in M_{p,q}(\mathbb{B}(Z_1, Z_2)_N)$ is a matrix of adjointables which are also left M -module maps, write $f \otimes g$ for $[f_{ij} \otimes g_{kl}] \in M_{mp,nq}(\mathbb{B}(Y_1 \otimes_\theta Z_1, Y_2 \otimes_\theta Z_2))$. Then $\|f \otimes g\|_{cb} \leq \|f\|_{cb} \|g\|_{cb}$, where the ‘subscript cb ’ refers to the norm discussed above the theorem. Further, $f \otimes g = \tilde{g} \otimes \tilde{f}$ for any $f \in \mathbb{B}(Y_1, Y_2)_M$ and $g \in \mathbb{B}(Z_1, Z_2)_N$.*

Proof. Suppose that $f \in \mathbb{B}(Y_1, Y_2)_M$ and $g \in \mathbb{B}(Z_1, Z_2)_N$. Since $g : Z_1 \rightarrow Z_2$ is a left M -module map, \tilde{g} is a right M -module map. Thus we can define $f \otimes_M^{\sigma_h} g : Y_1 \otimes_M^{\sigma_h} Z_1 \rightarrow Y_2 \otimes_M^{\sigma_h} Z_2$, and $\tilde{g} \otimes_M^{\sigma_h} \tilde{f} : \tilde{Z}_2 \otimes_M^{\sigma_h} \tilde{Y}_2 \rightarrow \tilde{Z}_1 \otimes_M^{\sigma_h} \tilde{Y}_1$. By Corollary 2.4 in [6], $f \otimes g : Y_1 \otimes_\theta Z_1 \rightarrow Y_2 \otimes_\theta Z_2$ is a completely bounded weak* continuous right N -module map, with $\|f \otimes g\|_{cb} \leq \|f\|_{cb} \|g\|_{cb}$. That $f \otimes g$ is adjointable follows from the fact that it is weak* continuous (by the remark a couple of paragraphs above Definition 1.3). Alternatively, for $y \in Y_1, z \in Z_1, w \in \tilde{Z}_2, x \in \tilde{Y}_2$ we have

$$\begin{aligned} (w \otimes x, f(y) \otimes g(z))_N &= (w, \theta((x, f(y))_M) g(z))_N \\ &= (w \theta((\tilde{f}(x), y)_M), g(z))_N \\ &= (\tilde{g}(w \theta((\tilde{f}(x), y)_M)), z)_N \\ &= (\tilde{g}(w), \theta((\tilde{f}(x), y)_M) z)_N \\ &= (\tilde{g}(w) \otimes \tilde{f}(x), y \otimes z)_N, \end{aligned}$$

which also yields the last statement.

Next, let $f = [f_{ij}], g = [g_{kl}]$ be as in the statement. By a two step method we may assume that $f = I$ or $g = I$. If $g = I$ the norm inequality we want follows from the case in the last paragraph, Lemma 2.2, and because

$$M_{m,n}(\mathbb{B}(Y_1 \otimes_\theta Z_1, Y_2 \otimes_\theta Z_2)) \cong \mathbb{B}(C_n(Y_1 \otimes_\theta Z_1), C_m(Y_2 \otimes_\theta Z_2))$$

which may be viewed as $\mathbb{B}(C_n(Y_1) \otimes_\theta Z_1, C_m(Y_2) \otimes_\theta Z_2)$. Thus $[f_{ij} \otimes 1]$ may be regarded as the map $h \otimes I$ on $C_n(Y_1) \otimes_\theta Z_1$, where h is the map $C_n(Y_1) \rightarrow C_n(Y_2)$ associated with $[f_{ij}]$ as in the discussion above Lemma 2.2. Thus

$$\|[f_{ij} \otimes 1]\|_{cb} = \|h \otimes I\|_{\mathbb{B}(C_n(Y_1) \otimes_\theta Z_1, C_m(Y_2) \otimes_\theta Z_2)} \leq \|h\|_{cb} = \|f\|_{cb}.$$

If $f = I$, the norm inequality we want follows by a standard trick (which could also have been used to give an alternative proof of the previous computation). If $y = [y_{ij}] \in M_p(Y)$ and $z = [z_{ij}] \in M_p(Z)$ then

$$\sum_r y_{ir} \otimes g_{kl}(z_{rj}) = \lim_\alpha \sum_{r,k} y_{ir} \otimes g_{kl}(z_k^\alpha)(w_k^\alpha, z_{rj})$$

where $\phi_\alpha(z) = [(w_k^\alpha, z)]$ and $\psi_\alpha([b_k]) = \sum_k z_k^\alpha b_k$ are the maps in Definition 1.1, but for Z in place of Y . It follows that the norm of $[\sum_r y_{ir} \otimes g_{kl}(z_{rj})]$ is dominated by $\sup_\alpha \|[g_{kl}]\|_{cb} \|y\| \|z\|$. That $\|I \otimes g\|_{cb} \leq \|g\|_{cb}$ follows easily from this if we use [6, Corollary 2.8]. \square

Remark. A similar result holds for rigged modules over approximately unital operator algebras. These matricial versions of the functoriality of the tensor product will be used in [5].

The w^* -rigged interior tensor product is *projective*. This is because of the following:

Proposition 2.4. *The normal module Haagerup tensor product is projective: If $u : Y_1 \rightarrow Y_2$ is a weak* continuous M -module complete quotient map between right dual operator M -modules, and $v : Z_1 \rightarrow Y_2$ is a weak* continuous M -module complete quotient map between left dual operator M -modules, then $u \otimes v : Y_1 \otimes_M^{\sigma_h} Z_1 \rightarrow Y_2 \otimes_M^{\sigma_h} Z_2$ is a weak* continuous complete quotient map.*

Proof. By functoriality of $\otimes_M^{\sigma_h}$ (see [6, Corollary 2.4]) we have that $u \otimes v$ is a weak* continuous complete contraction. If $z \in \text{Ball}(Y_2 \otimes_M^{\sigma_h} Z_2)$, then by [6, Corollary 2.8] z is weak* approximable by a net $z_t \in \text{Ball}(Y_2 \otimes_{hM} Z_2)$. We may assume that $\|z_t\| < 1$ for all t . By projectivity of \otimes_{hM} (or of \otimes_h) there exist $w_t \in \text{Ball}(Y_1 \otimes_{hM} Z_1)$ with $(u \otimes v)(w_t) = z_t$. Suppose that $w_{t_\nu} \rightarrow w \in \text{Ball}(Y_1 \otimes_M^{\sigma_h} Z_1)$, then clearly $(u \otimes v)(w) = z$. So $u \otimes v$ is a quotient map. A similar argument works at the matrix levels using [6, Corollary 2.8]. \square

Unlike the module Haagerup tensor product over a C^* -algebra (see [9, Theorem 3.6.5 (2)]), the normal module Haagerup tensor product $\otimes_M^{\sigma_h}$ need not be ‘injective’ for general dual operator modules if M is a W^* -algebra. However by functoriality it is easy to see that we will have such injectivity for this tensor product for w^* -orthogonally complemented submodules (see the last section of [5]) of w^* -rigged modules, even if M is a dual operator algebra. For example if Y is a w^* -orthogonally complemented submodule of a w^* -rigged module W over M , and if i and P are the associated inclusion and projection maps, and if Z is a right M - N -correspondence, then $P \otimes I_Z$ and $i \otimes I_Z$ are completely contractive adjointable maps composing to the identity on $Y \otimes_\theta Z$. So $Y \otimes_\theta Z$ is weak* homeomorphically completely isometrically M -module isomorphic to a w^* -orthogonally complemented submodule of $W \otimes_\theta Z$. A similar statement may be made for an appropriately complemented M - N -‘subcorrespondence’ of Z .

This injectivity of the weak* interior tensor product will work for any weak* closed submodules of w^* -rigged modules over a W^* -algebra (that is, W^* -modules), because such are automatically w^* -orthogonally complemented [4]. It follows that the weak* interior tensor with a right correspondence is ‘exact’ on the category of W^* -modules. Thus given an exact sequence of W^* -modules over a W^* -algebra M ,

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

with the first of these adjointable morphisms completely isometric and the second a complete quotient map, and given a right M - N -correspondence Z , we get an exact sequence

$$0 \longrightarrow D \otimes_\theta Z \longrightarrow E \otimes_\theta Z \longrightarrow F \otimes_\theta Z \longrightarrow 0$$

of the same kind. Indeed this all follows from the ‘commutation with direct sums’ property at the end of [7, Section 3]. It might be interesting to investigate such exactness of the interior tensor with a correspondence on the category of w^* -rigged modules.

2.3. HOM-tensor relations. (See [4, Theorem 3.6] for the self-adjoint variant of these.) In our context, the HOM spaces will be the spaces $\mathbb{B}(-, -)$ of all weak* continuous completely bounded module maps.

Lemma 2.5. *If Y is a right w^* -rigged module over M , and Z is a left (resp. right) dual operator module over M , then $Y \otimes_M^{\sigma^h} Z \cong \mathbb{B}_M(\tilde{Y}, Z)$ (resp. $Z \otimes_M^{\sigma^h} \tilde{Y} \cong \mathbb{B}(Y, Z)_M$) completely isometrically and weak* homeomorphically.*

Proof. The respectively case is [7, Theorem 3.5]. By the ‘other-handed variant’ of [7, Theorem 3.5] we have $CB_M^{\sigma^h}(\tilde{Y}, Z) \cong \tilde{Y} \otimes_M^{\sigma^h} Z = Y \otimes_M^{\sigma^h} Z$. \square

Theorem 2.6. *Let M and N be dual operator algebras. We have the following completely isometric identifications:*

- (1) $\mathbb{B}(Y \otimes_{\theta} Z, W)_N \cong \mathbb{B}(Y, \mathbb{B}(Z, W)_N)_M$, where Y is a right w^* -rigged module over M , Z is a right M - N correspondence, and W is a right dual operator module over N .
- (2) $\mathbb{B}(Y, (Z \otimes_N^{\sigma^h} W))_M \cong Z \otimes_N^{\sigma^h} \mathbb{B}(Y, W)_M$, where Y is a right w^* -rigged module over M , W is a dual operator N - M -bimodule and Z is a right dual operator N -module.
- (3) $\mathbb{B}_N(\mathbb{B}(Y, W)_M, X) \cong Y \otimes_M^{\sigma^h} \mathbb{B}_N(W, X)$, where Y is a right w^* -rigged module over M , X is a dual left N -operator module, and W is a left N - M -correspondence.
- (4) $\mathbb{B}_M(X, \mathbb{B}(Z, W)_N) \cong \mathbb{B}(Z, \mathbb{B}_M(X, W))_N$ where X , Z are left and right w^* -rigged modules over M and N respectively, and W is a dual operator M - N -bimodule.

Proof. The proofs all follow from Lemma 2.5, and Corollary 3.3 in [7], and the associativity of the normal module Haagerup tensor product, and the fact that $\tilde{\tilde{Y}} = Y$ for w^* -rigged modules. Since the proofs are all similar we just prove a couple of them. For (1) note that

$$\mathbb{B}_N(Y \otimes_{\theta} Z, W) \cong W \otimes_N^{\sigma^h} \widetilde{Y \otimes_{\theta} Z} \cong W \otimes_N^{\sigma^h} (\tilde{Z} \otimes_M^{\sigma^h} \tilde{Y}) \cong (W \otimes_N^{\sigma^h} \tilde{Z}) \otimes_M^{\sigma^h} \tilde{Y}$$

which is isomorphic to $\mathbb{B}(Z, W)_N \otimes_M^{\sigma^h} \tilde{Y} \cong \mathbb{B}(Y, \mathbb{B}(Z, W)_N)_M$. For (3),

$$\mathbb{B}_N(\mathbb{B}(Y, W)_M, X) \cong \mathbb{B}_N(W \otimes_M^{\sigma^h} \tilde{Y}, X) \cong (W \otimes_M^{\sigma^h} \tilde{Y}) \otimes_N^{\sigma^h} X \cong (Y \otimes_M^{\sigma^h} \tilde{W}) \otimes_N^{\sigma^h} X$$

which is isomorphic to $Y \otimes_M^{\sigma^h} (\tilde{W} \otimes_N^{\sigma^h} X) \cong Y \otimes_M^{\sigma^h} \mathbb{B}_N(W, X)$. \square

3. THE PICARD GROUP

We now discuss the Picard group of a dual operator algebra, following the route in [1]. Throughout this section A will be a dual operator algebra. We define the *weak Picard group* of A , denoted by $\text{Pic}_w(A)$, to be the collection of all A - A -bimodules implementing a weak* Morita equivalence of A with itself, with two such bimodules identified if they are completely isometrically isomorphic and weak* homeomorphic via an A - A -bimodule map. The multiplication on $\text{Pic}_w(A)$ is given by the module normal Haagerup tensor product $\otimes_A^{\sigma^h}$.

Any weak* continuous completely isometric automorphism θ of A defines a weak* Morita equivalence A - A -bimodule A_{θ} by ‘change of rings’ on the right. This is just A with the usual left module action, and with right module action $x \cdot a = x\theta(a)$.

Lemma 3.1. *The bimodule A_θ above is a weak* Morita equivalence bimodule for A , with ‘inverse bimodule’ $A_{\theta^{-1}}$.*

Proof. This follows using Definition 1.3 and Lemma 3.3 below, but we will give a more explicit proof. If A is a dual operator algebra, then we prove that $(A, A, A_\theta, A_{\theta^{-1}})$ is a weak* Morita context using Theorem 3.3 in [6]. Define a pairing $(\cdot) : A_\theta \times A_{\theta^{-1}} \rightarrow A$ taking $(a, a') \mapsto a\theta(a')$. It is easy to check that (\cdot) is a separately w^* -continuous completely contractive A -bimodule map which is balanced over A . Similarly, define another pairing $[\cdot] : A_{\theta^{-1}} \times A_\theta \rightarrow A$ taking $[a, a'] \mapsto a\theta^{-1}(a')$. Again it is easy to check that $[\cdot]$ is a separately w^* -continuous completely contractive A -module map which is balanced over A . It is simple algebra to check that $(x, y)x' = x[y, x']$ and $y'(x, y) = [y', x]y$. Checking the last assertion of Theorem 3.3 in [6] is also obvious. Hence $(A, A, A_\theta, A_{\theta^{-1}})$ is a weak* Morita context. \square

Let $\text{Aut}(A)$ denote the group of weak* continuous completely isometric automorphisms of A . For $\alpha, \beta \in \text{Aut}(A)$ let ${}_\alpha A_\beta$ denote A viewed as an A - A -bimodule with the left action $a \cdot x = \alpha(a)x$ and right action $x \cdot b = x\beta(b)$.

Lemma 3.2. *If $\alpha, \beta, \gamma \in \text{Aut}(A)$ then, ${}_\alpha A_\beta \cong {}_{\gamma\alpha} A_{\gamma\beta}$ completely A - A -isometrically.*

Proof. The map γ is the required isomorphism. \square

Lemma 3.3. *For $\theta_1, \theta_2 \in \text{Aut}(A)$, $A_{\theta_1} \otimes_A^{\sigma^h} A_{\theta_2} \cong A_{\theta_1\theta_2}$ completely A - A -isometrically.*

Proof. From Lemma 3.2, $A_{\theta_1} \otimes_A^{\sigma^h} A_{\theta_2} \cong {}_{\theta_1^{-1}} A \otimes_A^{\sigma^h} A_{\theta_2} \cong {}_{\theta_1^{-1}} A_{\theta_2} \cong A_{\theta_1\theta_2}$. \square

Proposition 3.4. *The collection $\{A_\theta : \theta \in \text{Aut}(A)\}$ constitutes a subgroup of $\text{Pic}_w(A)$, which is isomorphic to the group $\text{Aut}(A)$ of weak* continuous completely isometric automorphisms of A .*

Proof. This follows from the above lemma. \square

If X is a weak* Morita equivalence A - A -bimodule, and if $\theta \in \text{Aut}(A)$, then let X_θ be X with the same left module action, but with right module action changed to $x \cdot a = x\theta(a)$.

Lemma 3.5. *If X is a weak* Morita equivalence bimodule, then $X_\theta \cong X \otimes_A^{\sigma^h} A_\theta$ completely A - A -isometrically. Also, X_θ is a weak* Morita equivalence bimodule.*

Proof. The module action $(x, a) \mapsto xa$ is a completely contractive, separately weak* continuous balanced bilinear map, so by the universal property of the normal module Haagerup tensor product it induces a completely contractive weak* continuous linear map $m : X \otimes_A^{\sigma^h} A_\theta \rightarrow X_\theta$. There is a completely contractive inverse map $x \mapsto x \otimes 1$, so that m is a surjective complete isometry, and it is easily seen to be an A - A -bimodule map. If (A, A, X, Y) is a weak Morita context as in Definition 1.3 then by Lemma 3.1, and properties of the normal module Haagerup tensor product, $(A, A, X_{\theta, \theta^{-1}} Y)$ is a Morita context. To see this, note that by associativity of the normal module Haagerup tensor product, and some of the lemmas above in the present section,

$$(X \otimes_A^{\sigma^h} A_\theta) \otimes_A^{\sigma^h} (A_{\theta^{-1}} \otimes_A^{\sigma^h} Y) \cong X \otimes_A^{\sigma^h} A \otimes_A^{\sigma^h} Y \cong X \otimes_A^{\sigma^h} Y \cong A$$

completely isometrically and weak*-homeomorphically. Similarly

$$(A_{\theta^{-1}} \otimes_A^{\sigma^h} Y) \otimes_A^{\sigma^h} (X \otimes_A^{\sigma^h} A_\theta) \cong A_{\theta^{-1}} \otimes_A^{\sigma^h} A \otimes_A^{\sigma^h} A_\theta \cong A_{\theta^{-1}} \otimes_A^{\sigma^h} A_\theta \cong A$$

completely isometrically and weak*-homeomorphically. \square

An A - A -bimodule X will be called ‘symmetric’ if $ax = xa$ for all $a \in A$, $x \in X$.

Proposition 3.6. *For a commutative dual operator algebra A , $\text{Pic}_w(A)$ is a semidirect product of $\text{Aut}(A)$ and the subgroup of $\text{Pic}_w(A)$ consisting of symmetric equivalence bimodules. Thus, every $V \in \text{Pic}_w(A)$ equals X_θ , for a symmetric $X \in \text{Pic}_w(A)$, and some $\theta \in \text{Aut}(A)$.*

Proof. Suppose that X is any weak* Morita equivalence A - A -bimodule. Then any w^* -continuous right A -module map $T : X \rightarrow X$ is simply left multiplication by a fixed element of A . In fact we have $A \cong CB^\sigma(X)_A$, via a map $L : A \rightarrow CB(X)$ (see e.g. Theorem 3.6 in [6]). For fixed $a \in A$, the map $x \mapsto xa$ on X , is a w^* -continuous completely bounded A -module map with completely bounded norm $= \|a\|$. Hence by the above identification, there exists a unique $a' \in A$ such that $a'x = xa$ for all $x \in X$ and $\|a'\| = \|a\|$. Define $\theta(a) = a'$, then we claim that θ is a weak* continuous completely isometric unital automorphism of A . To see that θ is a homomorphism, let $a_1, a_2 \in A$ and let T, S and U be maps from X to X simply given by right multiplication with a_1, a_2 and a_1a_2 respectively. Let $\theta(a_1) = a'_1$, $\theta(a_2) = a'_2$ and $\theta(a_1a_2) = a'_3$. Since $U = ST$, we have $L(U) = L(T)L(S)$ (recall L is an anti-homomorphism, see e.g. Theorem 3.6 in [6]). This implies $a'_3 = a'_1a'_2$, that is, $\theta(a_1a_2) = \theta(a_1)\theta(a_2)$.

Note that

$$\|[\theta(a_{ij})x_{kl}]\| = \|[x_{kl}a_{ij}]\| \leq \|[x_{kl}]\| \|a_{ij}\|,$$

and so using the isomorphism $A \cong CB_A^\sigma(X)$ above we see that θ is completely contractive. That θ is completely isometric follows (e.g. by a similar argument for θ^{-1}). For the weak* continuity, note that if we have a bounded net $a_t \xrightarrow{w^*} a$ then $xa_t \xrightarrow{w^*} xa$ for all $x \in X$. Since the map L above is a weak* homeomorphism, we deduce that $\theta(a_t) \xrightarrow{w^*} \theta(a)$. That θ is surjective follows by symmetry.

Thus we have defined a surjective group homomorphism $\text{Pic}_w(A) \rightarrow \text{Aut}(A)$ taking $X \mapsto \theta$. To see that this does define a group homomorphism, let $X, Y \in \text{Pic}_w(A)$. Let $X \otimes_A^{\sigma^h} Y \mapsto \theta$ and $X \mapsto \theta_1, Y \mapsto \theta_2$ under the above identification. We need to show that $\theta = \theta_1\theta_2$. Let $a \in A$ and $\theta(a) = a'$, $\theta_2(a) = a_2$ and $\theta_1(a_2) = a_1$. We need to show that $a' = a_1$. Consider a rank one tensor $x \otimes y \in X \otimes_A^{\sigma^h} Y$ for $x \in X$ and $y \in Y$. Then $a'x \otimes y = x \otimes ya$, $a_1x = xa_2$ and $a_2y = ya$. We have $a'(x \otimes y) = a'x \otimes y$ and

$$a_1(x \otimes y) = a_1x \otimes y = xa_2 \otimes y = x \otimes a_2y = x \otimes ya.$$

Since finite rank tensors are weak* dense in $X \otimes_A^{\sigma^h} Y$, we have $a'z = a_1z$ for all $z \in X \otimes_A^{\sigma^h} Y$. This implies $a' = a_1$ which proves the required assertion.

From Lemma 3.1, for $\theta \in \text{Aut}(A)$ we have $A_\theta \in \text{Pic}_w(A)$, hence the above homomorphism has a 1-sided inverse $\text{Aut}(A) \rightarrow \text{Pic}_w(A)$. The above homomorphism restricted to modules of the form A_θ for $\theta \in \text{Aut}(A)$ is the identity map. That is, for $\theta \in \text{Aut}(A)$ the above homomorphism takes the weak* Morita equivalence bimodule A_θ to θ . Moreover the kernel of the homomorphism equals the symmetric equivalence bimodules. This proves the ‘semidirect product’ statement. For the last assertion, note that $X = (X_{\theta^{-1}})_\theta$. From the above, $x\theta^{-1}(a) = \theta(\theta^{-1}(a))x = ax$, which proves that $X_{\theta^{-1}}$ is symmetric. \square

Remark. Similar results will hold for strong Morita equivalence bimodules in the sense of [11] over a norm closed operator algebra A , and their associated Picard group.

Thus we may assume henceforth that X is symmetric, if A is a commutative dual operator algebra.

Proposition 3.7. *Suppose that we have a weak* Morita context (A, A, X, Y) , with A a weak* closed subalgebra of a commutative von Neumann algebra M . Suppose that A generates M as a von Neumann algebra. Then every symmetric weak equivalence A - A -bimodule is completely isometrically A - A -isomorphic to a weak* closed A - A -subbimodule of M .*

Proof. From Theorem 5.5 in [6], X dilates to a weak Morita equivalence M - M -bimodule $W = M \otimes_A^{\sigma_h} X$. Let (M, M, W, Z) be the corresponding W^* -Morita context. From Theorem 3.5 in [18], W contains X completely isometrically as a weak* closed A -submodule; and indeed it is clear that this is as a sub-bimodule over A . It is helpful to consider the inclusion

$$\begin{bmatrix} A & X \\ Y & A \end{bmatrix} \subset \begin{bmatrix} M & W \\ Z & M \end{bmatrix}$$

of linking algebras. If X is symmetric, then since $W = M \otimes_A^{\sigma_h} X$ we have $wa = aw$ for all $w \in W, a \in A$. Similarly for $Z \cong Y \otimes_A^{\sigma_h} M$, we have $za = az$. Since $Z = W^*$, we have $wm^* = (mw^*)^* = (w^*m)^* = m^*w$. Therefore $xw = wx$ for all $w \in W, x \in M$. Thus W is a symmetric element of $\text{Pic}_w(M)$.

If M is a commutative von Neumann algebra, then it is well known that the Picard group of M is just $\text{Aut}(M)$, and M is the only symmetric element of $\text{Pic}_w(M)$. We include a quick proof of this for the reader's convenience. Indeed suppose that Z is a symmetric weak equivalence M - M -bimodule. Suppose that M is a von Neumann algebra in $B(H)$, and let $K = Z \otimes_{\theta} H$, the induced representation. Then $Z \subset B(H, K)$ is a WTRO, which is commutative in the sense of [9, Proposition 8.6.5], and hence (see e.g. the proof of the last cited result) Z contains a unitary u with $uu^* = I_K$ and $u^*u = I_H$. Also, the map $R : z \mapsto u^*z$ is a completely isometric right M -module map from Z onto M . That is, $Z = uM$. Note that if $\theta : M \rightarrow B(Z)$ is the left action of M on Z , then since Z is symmetric we have $\theta(a)(ub) = uba = uau^*(ub)$, for $a, b \in M$. That is, $\theta(a)$ corresponds to $a \mapsto uau^* \in B(K)$. Then $R(\theta(a)z) = u^*uau^*z = aR(z)$, so that R is a bimodule map. That is, $Z \cong M$ as equivalence M - M -bimodules. Hence the Picard group of M is just $\text{Aut}(M)$.

Putting the last two paragraphs together, we have proved that every symmetric A - A -bimodule 'is' a weak* closed A - A -subbimodule of M . \square

Corollary 3.8. *The weak Picard group of $H^\infty(\mathbb{D})$ is isomorphic to the group of conformal automorphisms of the disk.*

Proof. Since the monomial z generates $A = H^\infty(\mathbb{D})$ as a dual algebra, any automorphism θ of A defines a map τ on the disk by $\theta(f)(w) = f(\tau(w))$, so that $\tau = \theta(z) \in H^\infty$. By looking at θ^{-1} , it is easy to see that τ is a conformal map. Thus the group of conformal automorphisms of the disk is isomorphic to the group $\text{Aut}(A)$. We will be done by Proposition 3.6, if we can show that every symmetric equivalence A - A -bimodule is A - A -isomorphic to A .

Let $M = L^\infty(\mathbb{T})$, and let X be a symmetric equivalence A - A -bimodule. By Proposition 3.7, X can be taken to be a weak* closed A -submodule of M . By Beurling's theorem it follows that X is singly generated. Indeed there is a function $k \in X$, which is either a projection or a unitary in M , with X equal to the weak*

closure of Mk or Ak respectively [16]. If k were a nontrivial projection $p \in M$, then $XM \subset kM$, so that X does not generate Z as an M -module, which is a contradiction. So k is unitary, and hence, as in the proof of Proposition 3.7, $X \cong A$ as dual operator A - A -bimodules. \square

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REFERENCES

- [1] D. P. Blecher and K. Jarosz, *Isomorphisms of function modules and generalized approximation in modulus*, Trans. Amer. Math. Soc. **354** (2002), 3663–3701.
- [2] D. P. Blecher, *A generalization of Hilbert modules*, J. Funct. Anal. **136** (1996), 365–421.
- [3] D. P. Blecher, *A new approach to Hilbert C^* -modules*, Math. Ann. **307** (1997), 253–290.
- [4] D. P. Blecher, *On selfdual Hilbert modules*, in “Operator algebras and their applications”, pp. 65–80, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997.
- [5] D. P. Blecher, *Rigged modules II: multipliers and duality*, Preprint 2016, to appear Studia Math.
- [6] D. P. Blecher and U. Kashyap, *Morita equivalence of dual operator algebras*, J. Pure and Applied Algebra **212** (2008), 2401–2412.
- [7] D. P. Blecher and U. Kashyap, *A characterization and a generalization of W^* -modules*, Trans. Amer. Soc. **363** (2011), 345–363.
- [8] D. P. Blecher and J. Kraus, *On a generalization of W^* -modules*, Banach Center Publications **91** (2010), 77–86.
- [9] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs, Oxford Univ. Press, Oxford, 2004.
- [10] D. P. Blecher and B. Magajna, *Duality and operator algebras: automatic weak continuity and applications*, J. Funct. Anal. **224** (2005), 386–407.
- [11] D. P. Blecher, P. S. Muhly, and V. I. Paulsen, *Categories of operator modules (Morita equivalence and projective modules)*, Mem. Amer. Math. Soc. **143** (2000), no. 681.
- [12] E. G. Effros and Z.-J. Ruan, *Representations of operator bimodules and their applications*, J. Operator Theory **19** (1988), 137–157.
- [13] E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Mathematical Society Monographs, New Series, 23, The Clarendon Press, Oxford University Press, New York, 2000.
- [14] G. Eleftherakis, *Some notes on Morita equivalence of operator algebras*, Serdica Math. J. **41** (2015), 117–128.
- [15] G. K. Eleftherakis and V. I. Paulsen, *Stably isomorphic dual operator algebras*, Math. Ann. **341** (2008), 99–112.
- [16] T. W. Gamelin, *Uniform algebras*, Prentice Hall (1969).
- [17] U. Kashyap, *Morita equivalence of dual operator algebras*, Ph. D. thesis (University of Houston), December 2008.
- [18] U. Kashyap, *A Morita theorem for dual operator algebras*, J. Funct. Anal. **256** (2009), 3545–3567.
- [19] E. C. Lance, *Hilbert C -modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, 210. Cambridge University Press, Cambridge, 1995.
- [20] B. Mesland, *Unbounded bivariant K -theory and correspondences in noncommutative geometry*, J. Reine Angew. Math. **691** (2014), 101–172.
- [21] W. L. Paschke, *Inner product modules over B^* -algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.
- [22] M. A. Rieffel, *Morita equivalence for C^* -algebras and W^* -algebras*, J. Pure Appl. Algebra **5** (1974), 51–96.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-3008
E-mail address: `dblechter@math.uh.edu`

DEPARTMENT OF STEM, REGIS COLLEGE, WESTON, MA 02493
E-mail address: `upasana.kashyap@regiscollege.edu`